Short note about the calculation of Jacobian and Inverse Jacobian functions, that are needed for the design of Cauer (= elliptical) filters.

In the following, parts from the book

HANDBOOK OF MATHEMATICAL FUNCTIONS
WITH FORMULAS, GRAPHS, AND MATHEMATICAL TABLES
Edited by
Milton Abramowitz and Irene A. Stegun
Dover Publications, Inc. (1964-72)
9th edition, 1972

concerning the use of the Arithmetic-Geometric Mean (A.G.M.) for calculations of the Jacobian Functions have been copied and ordered.

Next to that is described how the A.G.M. can be used to find Inverse Jacobian Functions.
This is my solution, which I have not found somewhere else. The $\mathrm{sn}^{-1}(u \mid m)$ calculation has been implemented in `Hs_cauer.m`.

More information about the A.G.M. and the Elliptic Integral can be found on the MathWorld pages, a.o. see
http://mathworld.wolfram.com/Arithmetic-GeometricMean.html
(by Eric W. Weisstein. "Arithmetic-Geometric Mean." From *MathWorld*--A Wolfram Web)

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16.4. Calculation of the Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

For the A.G.M. scale see **17.6**.

Inserted from pg. 598

17.6. The Process of the Arithmetic-Geometric Mean

Starting with a given number triple (a_0, b_0, c_0) we proceed to determine number triples $(a_1, b_1, c_1), (a_2, b_2, c_2), \cdots, (a_N, b_N, c_N)$ according to the following scheme of arithmetic and geometric means

a_0	b_0	c_0
$a_1 = \frac{a_0 + b_0}{2}$	$b_1 = \sqrt{a_0 b_0}$	$c_1 = \frac{a_0 - b_0}{2}$
$a_2 = \frac{a_1 + b_1}{2}$	$b_2 = \sqrt{a_1 b_1}$	$c_2 = \frac{a_1 - b_1}{2}$
$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$
$a_N = \frac{a_{N-1} + b_{N-1}}{2}$	$b_N = \sqrt{a_{N-1} b_{N-1}}$	$c_N = \frac{a_{N-1} - b_{N-1}}{2}$

We stop at the *N*th step when $a_N = b_N$, i.e., when $c_N = 0$ to the degree of accuracy to which the numbers are required.

To calculate $\operatorname{sn}(u \mid m)$, $\operatorname{cn}(u \mid m)$ and $\operatorname{dn}(u \mid m)$, form the scale by starting with

16.4.1 $$a_0 = 1,\ b_0 = \sqrt{m_1},\ c_0 = \sqrt{m}$$ (**Note:** $m = k^2$, $m_1 = 1 - k^2$).

terminating at the step *N* when c_N is negligible to the accuracy required. Find φ_N where

16.4.2 $$\varphi_N = 2^N a_N u$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \cdots, \varphi_1, \varphi_0$ from the recurrence relation

16.4.3 $$\sin(2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n$$

(**Note:** thus $\varphi_{n-1} = \dfrac{\sin^{-1}\left(\dfrac{c_n}{a_n} \sin \varphi_n\right) + \varphi_n}{2}$)

Then

16.4.4 $$\operatorname{sn}(u\,|\,m)=\sin\varphi_0\,,\qquad \operatorname{cn}(u\,|\,m)=\cos\varphi_0\,,\qquad \text{and}\qquad \operatorname{dn}(u\,|\,m)=\frac{\cos\varphi_0}{\sin(\varphi_1-\varphi_0)}$$

From these all the other functions can be determined.

etc., etc.

Huib: $\operatorname{sc}(u\,|\,m)=\tan\varphi_0$

Calculation of the Inverse Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

First the derivation of $\operatorname{sn}^{-1}(u\,|\,m)$, so

Let $\operatorname{sn}(u\,|\,m)=\mathrm{SN}$, then find u when m is known using the A.G.M.

Solution:

Construct the number triplets $(a_0,b_0,c_0),\cdots,(a_N,b_N,c_N)$ as described before in **17.6**, with the same start values as listed in **16.4.1**

To find u, calculate

$$\varphi_0=\sin^{-1}\left(\,|\mathrm{SN}|\,\right)$$

and then compute successively $\varphi_1,\ \varphi_2,\cdots,\ \varphi_{N-1},\ \varphi_N$ from the recurrence relation

$$\varphi_n=\tan^{-1}\left(\frac{\sin(2\varphi_{n-1})}{\frac{c_n}{a_n}+\cos(2\varphi_{n-1})}\right)$$

in which $\left(2\varphi_{n-1}-\frac{\pi}{2}\right)\ \le\ (\varphi_n+q2\pi)\ \le\ 2\varphi_{n-1}$ with $q=0$ or $1,2,\ldots$

Finally $$u=\operatorname{sn}^{-1}(u\,|\,m)=\ \operatorname{sign}(\mathrm{SN})\frac{\varphi_N}{2^N a_N}$$

The following Table lists the procedure to find u for a given $\operatorname{cn}^{-1}(u\,|\,m)$, $dn^{-1}(u\,|\,m)$ and $sc^{-1}(u\,|\,m)$.

given that	$\text{cn}(u\mid m)=\text{CN}$	$\text{dn}(u\mid m)=\text{DN}$	$\text{sc}(u\mid m)=\text{SC}$
startvalue for φ_0	$\cos^{-1}(\text{CN})$	$\sin^{-1}\left(\frac{\sqrt{1-\text{DN}^2}}{k}\right)$	$\tan^{-1}(\lvert\text{SC}\rvert)$
resulting $u=$	$\frac{\varphi_N}{2^N a_N}$	$\frac{\varphi_N}{2^N a_N}$	$\text{sign}(\text{SC})\frac{\varphi_N}{2^N a_N}$

given the relation $\text{dn}(u\mid m)=\sqrt{1-k^2\text{sn}^2(u\mid m)}$