

Design Examples of a 6th Order Symmetrical Band Pass Filter

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This document describes the intermediate results obtained when designing a Wave Digital Filter Structure that shows a 6th order symmetrical transfer function in the discrete-time domain.

The resulting Wave Digital Band Pass Filter (wdbpf) has to conform to the following design constraints:

- it's pass band has to stretch from $\omega_L = \frac{0.13}{f_s}$ to $\omega_H = \frac{0.17}{f_s}$, with an equiripple transfer characteristic not exceeding 1 dB attenuation,
- the filter has to show transfer zeros in the lower and higher parts of the stop band at respectively $0, \frac{0.1}{f_s}$ against $\frac{0.2}{f_s}$ and $\frac{0.5}{f_s}$.

with f_s representing the sampling frequency.

First we have to translate these frequency points to the Continuous-Time Domain by doing a 'prewarp' using the inverse bilinear transformation

$$\omega_{CTD} = \pi \cdot \tan(f_{DTD} \cdot f_s)$$

where CTD denotes the Continuous-Time Domain and DTD indicates the Discrete-Time Domain. For our convenience we skip these domain indications as long as it is clear in which domain we are doing our computations. So, the parameters to be used for calculating our Continuous-Time filter (see Figure 1) now become

$$\begin{aligned} N &= 6 \\ r_p &= 1.00000 \text{ dB} \\ \omega_{s0} &= 0.00000 \text{ rad/s} \\ \omega_{s1} &= 0.32492 \text{ rad/s} \\ \omega_{s2} &= 0.72654 \text{ rad/s} \\ \omega_{s3} &= \infty \text{ rad/s} \end{aligned}$$

Using the Vlach approximation [Vla69] of the pass band, we can find the frequencies of the 0 dB points to show at

$$\begin{aligned} \omega_{p1} &= 0.44154 \text{ rad/s} \\ \omega_{p2} &= 0.50815 \text{ rad/s} \\ \omega_{p3} &= 0.58085 \text{ rad/s} \end{aligned}$$

We need to find the Belevitch polynomials $f(s)$, $g(s)$ and $h(s)$ in order to be able to compute the magnitude transfer function

$$H(s) = \frac{f(s)}{g(s)}$$

and to calculate the element values for the Jaumann structure.

However, although we can immediately write down $f(s)$ and $h(s)$ given the data above, this is not the case for the $g(s)$ polynomial. We have to find $g(s)$ by first constructing the polynomials $f f^*$ and $h h^*$, and deriving $g g^*$ from $g g^* = f f^* + h h^*$.

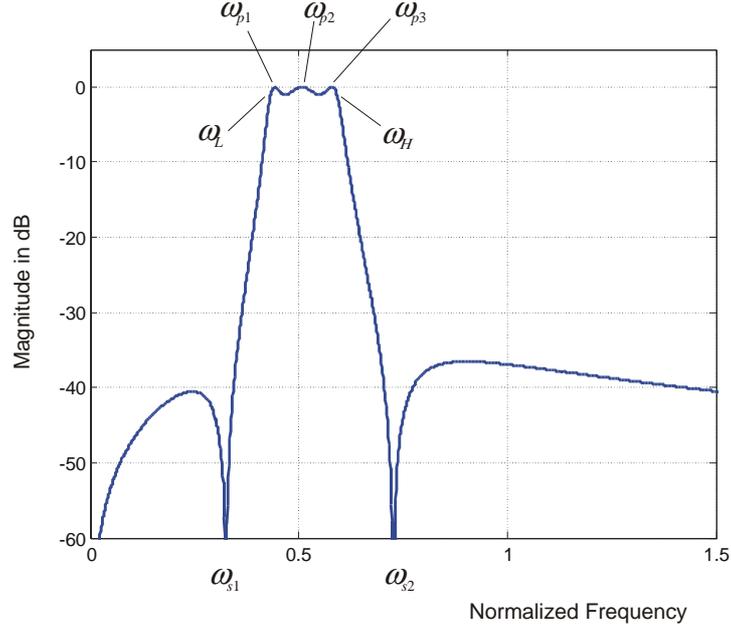


Figure 1: Magnitude Transfer Function in the Time Continuous Domain.

By substituting $j\omega$ by s , we obtain $g g_*$, from which we can finally construct $g(s)$, by noting that this polynomial has the same poles and zeros as $g g_*$, but only those in the left hand plane.

For the given 6th order filter, we write

$$f f^* = \omega^2 (\omega^2 - \omega_{s1}^2)^2 (\omega^2 - \omega_{s2}^2)^2$$

and

$$h h^* = \varepsilon^2 C_h^2 \cdot (\omega^2 - \omega_{p1}^2)^2 (\omega^2 - \omega_{p2}^2)^2 (\omega^2 - \omega_{p3}^2)^2$$

where

$$\varepsilon^2 = 0.25893 \quad (= 10^{rp/10} - 1)$$

$$C_h^2 = 21569 \quad (C_h^2 \text{ is chosen such that } [HH^*]_{\omega=\omega_{p2}} = 1)$$

This leads us to

$$f f^* = \omega^{10} - 1.2669\omega^8 + 0.51270\omega^6 - 0.0706\omega^4 + 0.0031056\omega^2, \text{ and}$$

$$h h^* = 5584.8\omega^{12} - 8830.3\omega^{10} + 5760.6\omega^8 - 1984.4\omega^6 + 380.67\omega^4 - 38.558\omega^2 + 1.6112$$

and so to

$$g g^* = 5584.8\omega^{12} - 8829.3\omega^{10} + 5759.4\omega^8 - 1983.9\omega^6 + 380.60\omega^4 - 38.555\omega^2 + 1.6112$$

from which we derive

$$g g_* = 5584.8s^{12} + 8829.3s^{10} + 5759.4s^8 + 1983.9s^6 + 380.60s^4 + 38.555s^2 + 1.6112$$

From this $g g_*$ we calculate the roots, and keep only those in the left hand plane to get $g(s)$:

$$g(s) = 74.731s^6 + 11.577s^5 + 59.971s^4 + 6.1295s^3 + 15.421s^2 + 0.76966s + 1.2693$$

We can again write down $f(s)$ and $h(s)$ immediately, viz.

$$f(s) = s(s^2 + \omega_{s1}^2)(s^2 + \omega_{s2}^2) = s^5 + 0.63344s^3 + 0.055728s$$

$$h(s) = \pm \varepsilon C_h \cdot (s^2 + \omega_{p1}^2)(s^2 + \omega_{p2}^2)(s^2 + \omega_{p3}^2)$$

Here we choose for the + sign, thus

$$h(s) = 74.731s^6 + 59.080s^4 + 15.188s^2 + 1.2693$$

Now that we know $H(s)$

$$H(s) = \frac{s^5 + 0.63344s^3 + 0.055728s}{74.731s^6 + 11.577s^5 + 59.971s^4 + 6.1295s^3 + 15.421s^2 + 0.76966s + 1.2693}$$

we can preview what the transfer function $H(z)$ in the discrete-time domain will look like. We can do this by either plotting the $|H(s)|$ transfer function with a bilinearly distorted horizontal axis ($\arctan \frac{\omega}{\pi}$ instead of ω), or by really computing the actual $H(z)$. Both methods result in the transfer function shown in Figure 2.

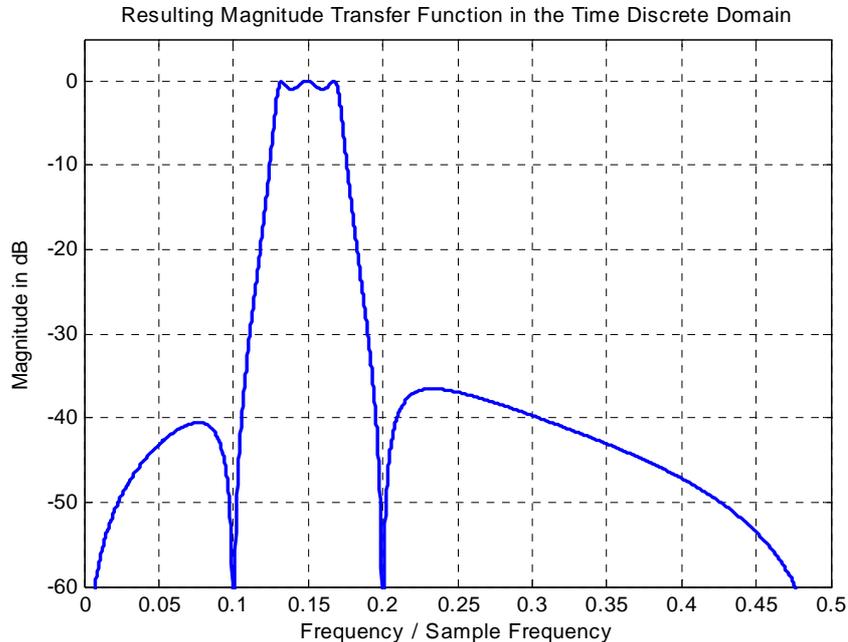


Figure 2: Preview of the expected discrete filter characteristic.

First, however, we will continue with the derivation of a Jaumann design:

Given the $f(s)$, $g(s)$ and $h(s)$ polynomials, we can derive the Tellegen polynomials A , H and C that are needed for determining the lossless impedances $Z_1(s)$ and $Z_2(s)$ in a Jaumann design.

Since $f(s)$ is an odd order function, we can conclude that also A will be odd:

$$A = \frac{\text{Odd}[g(s)]}{2} = 5.7887s^5 + 3.0648s^3 + 0.38483s$$

Then

$$C = \frac{\text{Even}[g(s)] - \text{Even}[h(s)]}{2} = 0.44505s^4 + 0.11606s^2$$

and

$$H = \frac{f(s)}{2} = 0.50000s^5 + 0.31672s^3 + 0.027864s$$

This means that we now do know $Z_1(s)$ and $Z_2(s)$, viz.

$$Z_1(s) = \frac{A + H}{C} = \frac{6.2887s^5 + 3.3815s^3 + 0.41269s}{0.44505s^4 + 0.11606s^2} = \frac{6.2887s^4 + 3.3815s^2 + 0.41269}{0.44505s^3 + 0.11606s}$$

$$Z_2(s) = \frac{A - H}{C} = \frac{5.2887s^5 + 2.7480s^3 + 0.35697s}{0.44505s^4 + 0.11606s^2} = \frac{5.2887s^4 + 2.7480s^2 + 0.35697}{0.44505s^3 + 0.11606s}$$

A closer look at $Z_2(s)$ reveals that nominator and denominator share a common factor $s^2 + 0.26078$, caused by both having roots at $\pm 0.510667j$. So $Z_2(s)$ can be reduced to

$$Z_2(s) = \frac{5.2887s^2 + 1.3688}{0.44505s}$$

The results thus are a 4th order impedance function for $Z_1(s)$ and a 2nd order impedance function for $Z_2(s)$.

By performing some simple one port network synthesis, we can find element values with which we should be able to construct a Jaumann structure in the Continuous- Time domain ($\frac{Z_1(s)}{2}$ and $2Z_2(s)$ needed).

L_2	23.76682	H
C_2	0.16257	F
L_{1a}	7.06517	H
C_{1b}	0.51111	F
L_{1c}	82.19461	H
C_{1d}	0.05134	F

A possible structure with the listed element values is given in Figure 3.

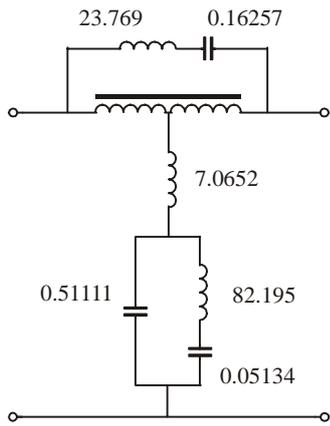


Figure 3: The Continuous-Time version of the 6th order Jaumann structure with lumped element values.

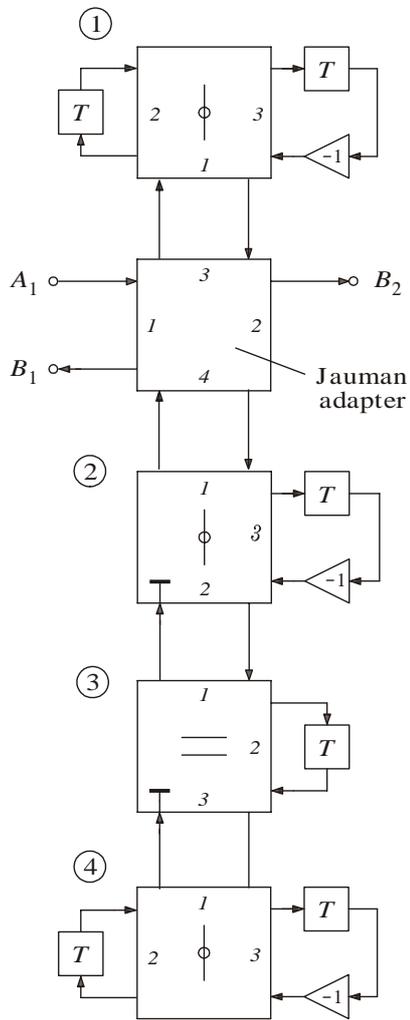


Figure 4: The translation of Figure 3 into a Jaumann structure with 3 port adapters.

Some implementations with discrete-time structures

In the previous section, we already derived the transfer function $H(s) = \frac{f(s)}{g(s)}$ in the s-domain and have found values describing the $f(s)$ and $g(s)$ polynomials.

We can, of course, also translate this $H(s)$ from the continuous-time domain into a $H(z)$ in the discrete-time domain using well known procedures. Since we implicitly used the inverse-bilinear transformation method to derive the specifications of the continuous-time filter, we will describe the results of the bilinear method here. It implies that we have to substitute

$$s = \frac{2}{T} \frac{z - 1}{z + 1} \quad (\text{we choose to use } T = 2 \text{ here})$$

in $f(s)$ and $g(s)$. We then find (Note: $b_3 = 0!$)

$$H(z) = \frac{0.0099439 - 0.022235z^{-1} + 0.019888z^{-2} - 0.019888z^{-4} + 0.022235z^{-5} - 0.0099439z^{-6}}{1.0000 - 3.3738z^{-1} + 6.5225z^{-2} - 7.6002z^{-3} + 6.0121z^{-4} - 2.8648z^{-5} + 0.78246z^{-6}}$$

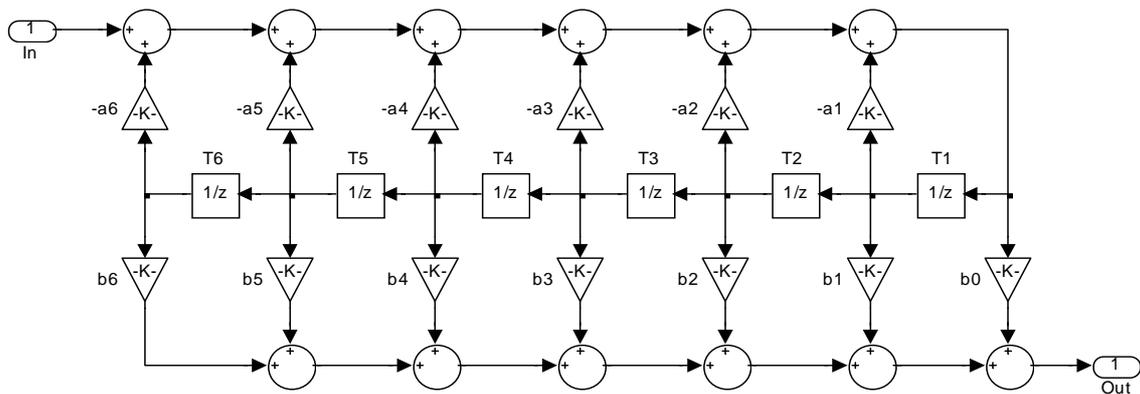


Figure 5: The band pass filter as a Direct-Form 2 recursive structure.

Second Order Sections

This $H(z)$ can easily be implemented as a Direct-Form 1 or Direct-Form 2 (see Figure 5) recursive structure. We will, however, show an alternative implementation with three Second-Order Direct-Form-1 sections. The multiplication factors to be used in the Second-Order sections can be calculated with MATLAB's `tf2sos`-function from the Signal Processing Toolbox. The general setup of a Second-Order section which can be described with

$$H(z)_{\text{2nd order section}} = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}$$

is given in Figure 6 (note the signs of the a_i -coefficients) and the resulting complete filter can be found in Figure 7. Compared with the structure in Figure 5 now only 10 (or 9) multipliers are needed instead of 13.

`tf2sos` gives us:

	b_0	b_1	b_2	a_1	a_2
section 1	1.0000	0.0000	-1.0000	-1.1038	0.8747
section 2	1.0000	-0.6180	1.0000	-0.9409	0.9442
section 3	1.0000	-1.6180	1.0000	-1.3291	0.9475

and an overall $g = 0.00994$.

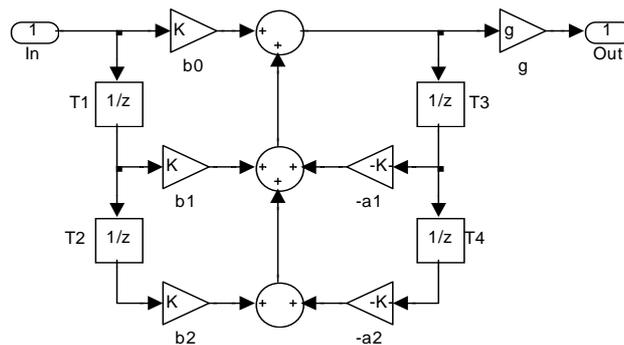


Figure 6: Second-Order Direct-Form 1 section.

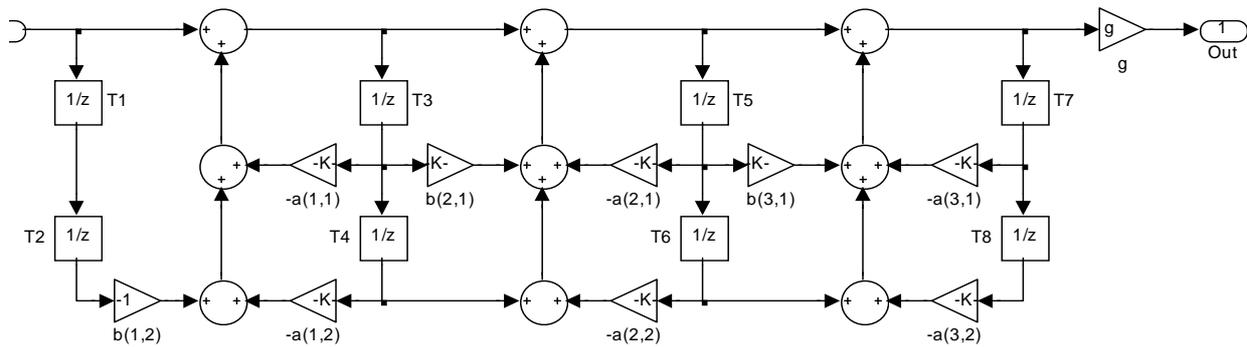


Figure 7: The 6th order bandpass transfer function as implemented with three Second-Order sections.

$$\begin{aligned}
v(1) &= -0.00114 \\
v(2) &= -0.00409 \\
v(3) &= 0.00144 \\
v(4) &= 0.00555 \\
v(5) &= 0.01194 \\
v(6) &= -0.01131 \\
v(7) &= -0.00994
\end{aligned}$$

The fact that $|k(i)| < 1$ for all i 's (1 to 6 here) ensures that this filter will be stable. The transfer characteristics of $H(z)$ and that resulting from a simulation of Figure 8, completely coincide and completely confirm to the expected transfer characteristic of Figure 2.

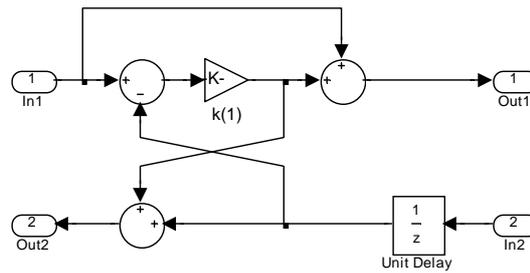


Figure 10: Version of Submodel Lattice1 using only one multiplier.

Instead of the lattice submodel of Figure 9, it is also possible to use the circuit of Figure 10 which uses only one multiplier. Although it can be derived that the transfer function (all-pass) from nodes In1 to Out2 for both submodels is the same, the transfer functions from In1 to Out1 do differ, being a factor $(1 + k_i)$ higher for the One multiplier version. This necessitates that the summing factors $v(1)$ upto $v(6)$ should be adapted according to

$$v'(n) = \frac{v(n)}{\prod_{m=n}^6 [1 + k(m)]} \quad n = [1, 6]$$

The result now will be

$$\begin{aligned}
v(1) &= -0.00239 \\
v(2) &= -0.00338 \\
v(3) &= 0.00236 \\
v(4) &= 0.00375 \\
v(5) &= 0.01595 \\
v(6) &= -0.00635 \\
v(7) &= -0.00994
\end{aligned}$$

Lattice Wave Digital Structure

In this section an implementation based on the subtraction of two all-pass function will be described [Gaz85], [Wan99]. Each all-pass function is built as cascaded Second-order Richards' structures, if needed –in case of an odd function– augmented with a First-order Richards' structure. What exactly is needed here can be derived from the Tellegen polynomials A , H and C that we found before, or alternatively from the Jaumann impedances $Z_1(s)$ and $Z_2(s)$. First, we have to find L and C element values that we can think of to be connected to each other using circulators. Next, these element values will be used for calculating the multiplication factors in the Richards' structures.

A block diagram of the discrete-time structure that we will use is given in Figure 11, with an expanded view of the 2nd order sections using 2 port adapters in Figures 12 and 13.

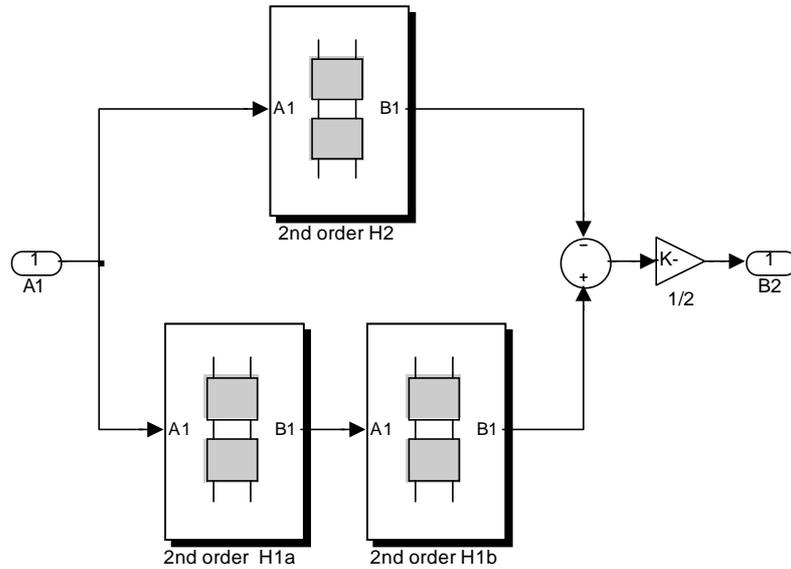


Figure 11: Lattice Wave Digital Filter.

So we start with writing down the reflection functions of the two all-pass sections:

$$\rho_1(s) = \frac{Z_1(s) - 1}{Z_1(s) + 1} = \frac{A + H - C}{A + H + C}$$

$$\rho_2(s) = \frac{Z_2(s) - 1}{Z_2(s) + 1} = \frac{A - H - C}{A - H + C}$$

Substituting the A , H and C that we found before, gives us

$$\rho_1(s) = \frac{6.2887s^4 - 0.44505s^3 + 3.3815s^2 - 0.11606s + 0.41269}{6.2887s^4 + 0.44505s^3 + 3.3815s^2 + 0.11606s + 0.41269}$$

and

$$\rho_2(s) = \frac{5.2887s^4 - 0.44505s^3 + 2.7480s^2 - 0.11606s + 0.35697}{5.2887s^4 + 0.44505s^3 + 2.7480s^2 + 0.11606s + 0.35697}$$

Here again, nominator and denominator of $\rho_2(s)$ share a common factor $s^2 + 0.26078$, leaving us

$$\rho_2(s) = \frac{5.2887s^2 - 0.44505s + 1.3688}{5.2887s^2 - 0.44505s + 1.3688}$$

Now, we will have to rewrite $\rho_1(s)$ and $\rho_2(s)$ as cascaded 2nd order sections. Observe that $\rho_2(s)$ already is in the correct form, so we only need to split $\rho_1(s)$, e.g. by combining complex conjugated roots:

$$\rho_1(s) = \frac{s^2 - 0.032069s + 0.18871}{s^2 + 0.032069s + 0.18871} \cdot \frac{s^2 - 0.038700s + 0.34775}{s^2 + 0.038700s + 0.34775}$$

Each 2nd order term describes an LC-series resonator circuit, for which we can immediately find the corresponding L and C element values:

$$\text{if } \rho(s) = \frac{s^2 - a_1s + a_2}{s^2 + a_1s + a_2}, \text{ then } Z(s) = \frac{1 + \rho(s)}{1 - \rho(s)}, \text{ so } L_s = \frac{1}{a_1} \text{ and } C_s = \frac{a_1}{a_2}.$$

We thus can derive the following element values:

$$\text{for } \rho_1(s): \begin{array}{l} L_1a = 31.1823 \text{ H} \\ C_1a = 0.16994 \text{ F} \\ L_1b = 25.8396 \text{ H} \\ C_1b = 0.11129 \text{ F} \end{array}$$

and

$$\text{for } \rho_2(s): \begin{array}{l} L_2 = 11.8834 \text{ H} \\ C_2 = 0.32513 \text{ F} \end{array}$$

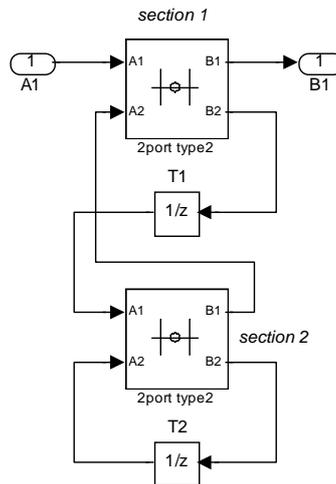


Figure 12: A Wave Digital Filter translation of a series resonance LC circuit using two 2 port adapters, where section 1 is characterized by $\alpha_{\text{section 1}}$ and section 2 by $\alpha_{\text{section 2}}$.

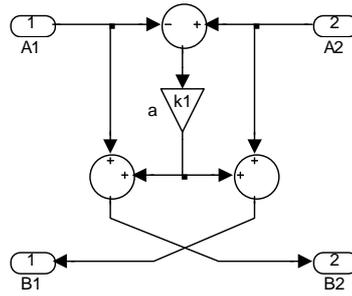


Figure 13: Implementation of the 2 port type 2 adapter.

For each 2nd order section, we can find the multiplication factors $\alpha_{\text{section 1}}$ and $\alpha_{\text{section 2}}$ to be used by a translation of the corresponding L_s and C_s values based on the bilinear transformation. It can be derived that

$$\alpha_{\text{section 1}} = \frac{C_s - L_s C_s - 1}{C_s + L_s C_s + 1}$$

and

$$\alpha_{\text{section 2}} = \frac{L_s C_s - 1}{L_s C_s + 1}$$

The final design parameters then become:

$$\text{for } \rho_1(s): \begin{array}{l} \alpha_{\text{section 1}} = -0.94746 \\ \alpha_{\text{section 2}} = 0.68250 \\ \alpha_{\text{section 1}} = -0.94417 \\ \alpha_{\text{section 2}} = 0.48395 \end{array}$$

and

$$\text{for } \rho_2(s): \begin{array}{l} \alpha_{\text{section 1}} = -0.87468 \\ \alpha_{\text{section 2}} = 0.58879 \end{array}$$

Again, the resulting magnitude transfer function is exactly identical to the theoretical $H(z)$.

Note

As pointed out in [Gaz85], it is of course possible to skip the intermediate calculation of L_s and C_s values, and instead use

$$\alpha_{\text{section 1}} = \frac{a_1 - a_2 - 1}{a_1 + a_2 + 1}$$

and

$$\alpha_{\text{section 2}} = \frac{1 - a_2}{1 + a_2}$$

Discussion

We derived continuous-time as well as discrete-time domain structures for a band pass filter that was requested to be symmetrical in the time-discrete domain. Crucial in the design, is the ability to compute an equiripple approximation of a user definable interval in the continuous-time domain with additional user definable zeros in the stopband, as described in [Vla69]. Resuming, we calculated continuous-time domain solutions for

- a ladder structure, and
- a Jaumann structure,

as well as discrete-time domain solutions for

- a wave digital Jaumann structure,
- a Direct-Form 2 recursive implementation,
- an IIR structure consisting of second-order sections,
- an IIR lattice structure, and finally
- a Lattice Wave Digital structure.

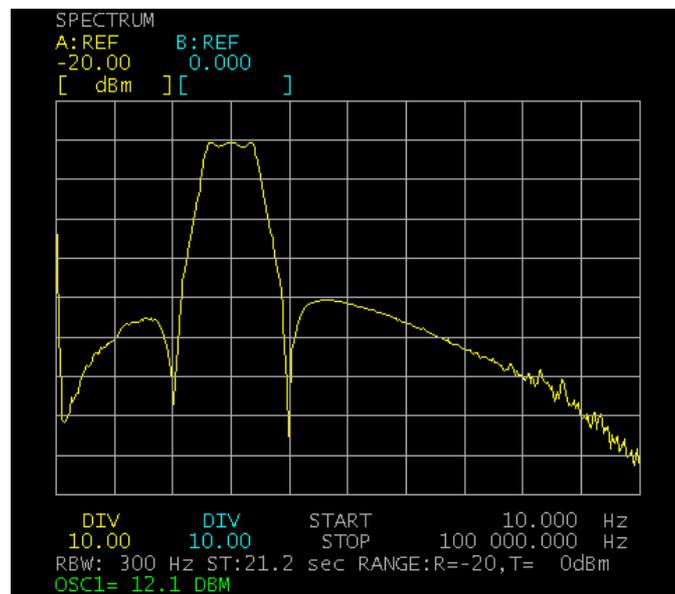


Figure 14: Measured spectrum plot of the 6th order lattice band pass structure implemented on an XC2V1000 FPGA on a Strathnuey card.

References

- [Gaz85] L. Gazsi, *Explicit Formulas for Lattice Wave Digital Filters*, IEEE Trans. on CAS, Vol. 32, pp. 68-88, Jan 1985.
- [Sha54] Ch.B. Sharpe, *A General Tchebycheff Rational Function*, Proc. IRE, Vol. 42, pp. 454-457 (1954).
- [Vla69] J. Vlach, *Computerized Approximation and Synthesis of Linear Networks*, John Wiley & Sons, Inc.(1969), SBN 471 90870 3.
- [Wan99] L. Wanhammer, *DSP Integrated Circuits*, Academic Press, 1999.

Er ontbreken nog de stukken waarin

- een tijd-continue ladder structuur berekend wordt,
- de wdf parameters van de Jaumann structuur berekend worden,
- een beschrijving van de implementatie m.b.v. A|RT C en/of the MATLAB/Xilinx System Generator op bijv. de Xtreme DSP Development Kit met z'n XC2V3000 fpga staat,
- tenslotte de meetresultaten getoond worden