On the following pages, a description is given of the equations, structures and symbols for the adaptors that are referred to in the (L)WDF Toolbox, viz.

- the three-port series adaptor,
- the three-port parallel adptor, and
- the two-port adaptor.

The reader is referred to the literature for the theory behind (Lattice) Wave Digital Filters.
Especially noteworthy are the articles from Fettweiss and Gazsi:

A. Fettweis
Wave Digital Filters: Theory and Practice (Invited Paper)
Proc. of the IEEE, Vol. 74, No, 2, Februari 1986

L. Gazsi
Explicit Formulas for Lattice Wave Digital Filters
IEEE Trans. on CAS, Vol. 32, pp. 68-88, Jan 1985

Huibert J. Lincklaen Arriëns
this version: January 2006

1 The Three-Port Series Adaptor

Given the equations directly following from Figure 1

$$U_1 + U_2 + U_3 = 0 \qquad \text{and} \qquad I_1 = I_2 = I_3 \tag{1}$$

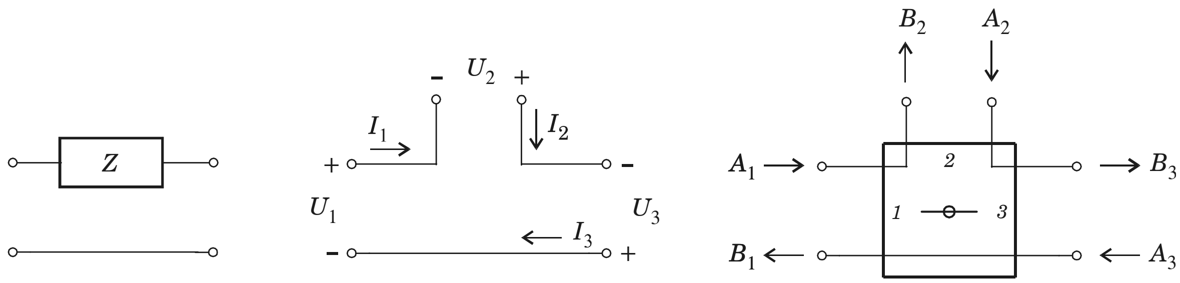

Figure 1: Definitions with respect to the three-port serial adapter

and the definitions of the voltage wave variables

$$A_k = U_k + I_k R_k \tag{2a}$$
$$B_k = U_k - I_k R_k \tag{2b}$$

which can be rewritten as

$$U_k = \frac{A_k + B_k}{2} \qquad \text{and} \qquad I_k = \frac{A_k - B_k}{2R_k} \qquad \text{with } k = 1, 2, 3 \tag{3}$$

we can perform the following derivations by combining equations (1) and (3)

$$A_1 + B_1 + A_2 + B_2 + A_3 + B_3 = 0 \tag{4}$$
$$\frac{A_1 - B_1}{R_1} = \frac{A_2 - B_2}{R_2} = \frac{A_3 - B_3}{R_3} \tag{5}$$

So, if we eliminate B_2 and B_3

$$B_2 = \frac{R_2}{R_1}(B_1 - A_1) - A_2 \qquad \text{and} \qquad B_3 = \frac{R_3}{R_1}(B_1 - A_1) - A_3 \tag{6}$$

we can express B_1 in terms of the A-inputs

$$\left(1 + \frac{R_2}{R_1} + \frac{R_3}{R_1}\right) B_1 = \left(-1 + \frac{R_2}{R_1} + \frac{R_3}{R_1}\right) A_1 - 2A_2 - 2A_3$$

$$B_1 = \left(1 - \frac{2R_1}{R_1 + R_2 + R_3}\right) A_1 - \frac{2R_1}{R_1 + R_2 + R_3} A_2 - \frac{2R_1}{R_1 + R_2 + R_3} A_3 \tag{7}$$

If we define

$$\frac{2R_1}{R_1 + R_2 + R_3} = \alpha_1 \tag{8}$$

then (7) reduces to

$$B_1 = (1 - \alpha_1)A_1 - \alpha_1 A_2 - \alpha_1 A_3 \tag{9}$$

More generally

$$B_k = A_k - \alpha_k (A_1 + A_2 + A_3) \qquad \text{where} \qquad \alpha_k = \frac{2R_k}{R_1 + R_2 + R_3} \tag{10}$$

It also follows that

$$\alpha_1 + \alpha_2 + \alpha_3 = 2 \qquad \text{while} \quad 0 \leq \alpha_k \leq 2 \tag{11}$$

We can write the wave equations in matrix notation: $\mathbf{B} = \hat{\mathbf{S}} \cdot \mathbf{A}$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 \\ -\alpha_2 & 1-\alpha_2 & -\alpha_2 \\ -\alpha_3 & -\alpha_3 & 1-\alpha_3 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{12}$$

If we now eliminate α_2 by realizing (see equation (11)) that

$$\alpha_2 = 2 - \alpha_1 - \alpha_3 \tag{13}$$

we obtain for the matrix $\hat{\mathbf{S}}$

$$\hat{\mathbf{S}} = \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 \\ \alpha_1 + \alpha_3 - 2 & \alpha_1 + \alpha_3 - 1 & \alpha_1 + \alpha_3 - 2 \\ -\alpha_3 & -\alpha_3 & 1-\alpha_3 \end{bmatrix} \tag{14}$$

Such an adaptor can be denoted as a Three-Port Series adapter with Port 2 being *dependent*, more generally called an *unconstrained* three-port series adapter.
A possible realization using 2 (constant coefficient) multipliers, 4 adders and 1 sign-invertor is given in Figure 2. The seriesl adaptor can be used to connect a single component in a series arm of a ladder structure, but also to connect the (translated) inductor and capacitor of a series resonator. A component determines the R-value of its port, while all three R-values are needed to obtain the multiplier constants α_1 and α_3.

Now if, e.g. in a ladder structure, we connect two adaptors (suppose Port 3 of this adaptor to Port 1 of a next one), we can freely choose the resistance value for these ports.

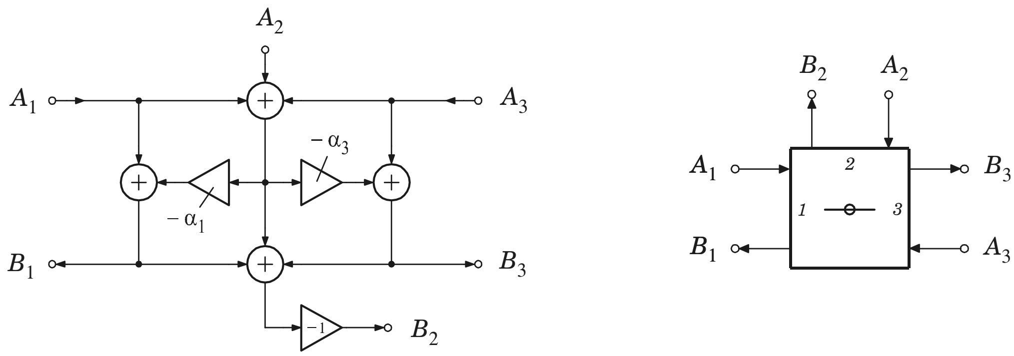

Figure 2: Signal-flow diagram for realizing the $\hat{\mathbf{S}}$-matrix of Equation 14 and its commonly used symbol.

Suppose, we choose for the first adaptor

$$R_3 = R_1 + R_2 \tag{15}$$

then we get

$$\alpha_3 = 1 \tag{16}$$

and we find that B_3 becomes independent of A_3,
as can be seen from

$$\hat{\mathbf{S}} = \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 \\ \alpha_1 - 1 & \alpha_1 & \alpha_1 - 1 \\ -1 & -1 & 0 \end{bmatrix} \tag{17}$$

In this case, we obtain a Three-Port Series adaptor with *reflection-free* Port 3 and Port 2 being dependent, which can be indicated as a *constrained* three-port series adaptor.
Figure 3 shows a possible realization and its commonly used symbol.

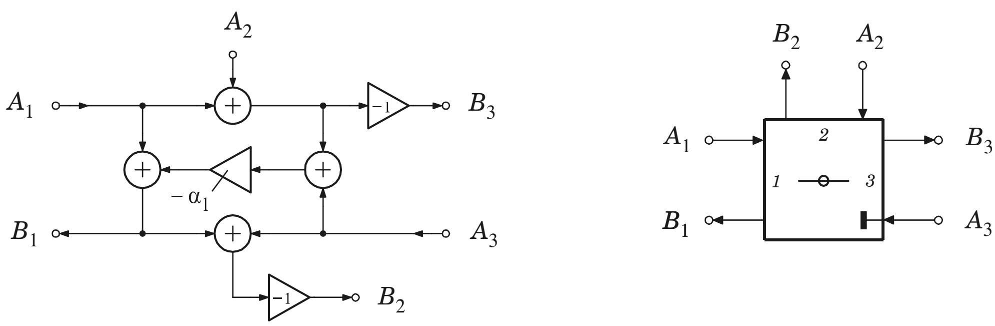

Figure 3: Possible structure for realizing the $\hat{\mathbf{S}}$-matrix of Equation 17 and the symbol to represents it with.

2 The Three-Port Parallel Adaptor

Given the equations directly following from Figure 4

$$U_1 = U_2 = U_3 \quad \text{en} \quad I_1 + I_2 + I_3 = 0 \tag{18}$$

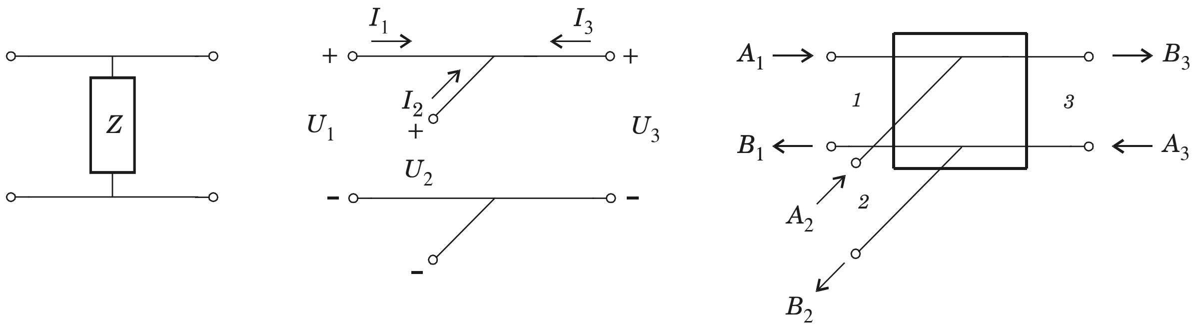

Figure 4: Definitions with respect to the three-port parallel adapter

and the definitions of the voltage wave variables

$$A_k = U_k + I_k R_k \tag{19a}$$
$$B_k = U_k - I_k R_k \tag{19b}$$

we find

$$U_k = \frac{A_k + B_k}{2} \quad \text{and} \quad I_k = \frac{A_k - B_k}{2R_k} \quad \text{with } k = 1, 2, 3 \tag{20}$$

While $g_k = \dfrac{1}{R_k}$, we can perform the following derivations by combining equations (18) and (20)

$$g_1 (A_1 - B_1) + g_2 (A_2 - B_2) + g_3 (A_3 - B_3) = 0 \tag{21}$$

$$A_1 + B_1 = A_2 + B_2 = A_3 + B_3 \tag{22}$$

So, if we eliminate B_2 and B_3

$$B_2 = A_1 + B_1 - A_2 \quad \text{and} \quad B_3 = A_1 + B_1 - A_3 \tag{23}$$

we can express B_1 in terms of the A-inputs

$$B_1 = \frac{g_1 - g_2 - g_3}{g_1 + g_2 + g_3} A_1 + \frac{2g_2}{g_1 + g_2 + g_3} A_2 + \frac{2g_3}{g_1 + g_2 + g_3} A_3 \tag{24}$$

If we define

$$\alpha_k = \frac{2g_k}{g_1 + g_2 + g_3} \tag{25}$$

then (24) reduces to

$$B_1 = (\alpha_1 - 1)\ A_1 + \alpha_2 A_2 + \alpha_3 A_3 \tag{26}$$

More generally

$$B_k = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 - A_k \qquad \text{where} \qquad \alpha_k = \frac{2g_k}{g_1 + g_2 + g_3} \tag{27}$$

It also follows that

$$\alpha_1 + \alpha_2 + \alpha_3 = 2 \qquad \text{while} \quad 0 \leq \alpha_k \leq 2 \tag{28}$$

We can write the wave equations in matrix notation: $\mathbf{B} = \hat{\mathbf{S}} \cdot \mathbf{A}$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 - 1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 - 1 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 - 1 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{29}$$

If we now eliminate α_2 by realizing (see equation (28)) that

$$\alpha_2 = 2 - \alpha_1 - \alpha_3 \tag{30}$$

we obtain for the matrix $\hat{\mathbf{S}}$

$$\hat{\mathbf{S}} = \begin{bmatrix} \alpha_1 - 1 & 2 - \alpha_1 - \alpha_3 & \alpha_3 \\ \alpha_1 & 1 - \alpha_1 - \alpha_3 & \alpha_3 \\ \alpha_1 & 2 - \alpha_1 - \alpha_3 & \alpha_3 - 1 \end{bmatrix} \tag{31}$$

Such an adaptor can be denoted as a Three-Port Parallel adaptor with Port 2 being *dependent*, more generally called an *unconstrained* three-port parallel adaptor.
A possible realization using 2 (constant coëfficiënt) multipliers, 4 adders and two sign-inversions is given in Figure 5. The parallel adaptor can be used to connect a single component in a shunt arm of a ladder structure, but also to connect the (translated) inductor and capacitor of a parallel resonator. A component determines the g-value of its port, while all three g-values are needed to obtain the multiplier constants α_1 and α_3.

Now if, e.g. in a ladder structure, we connect two adaptors (suppose Port 3 of this adaptor to Port 1 of a next one), we can freely choose the resistance value for these ports.
Suppose, we choose for the first adaptor

$$g_3 = g_1 + g_2 \tag{32}$$

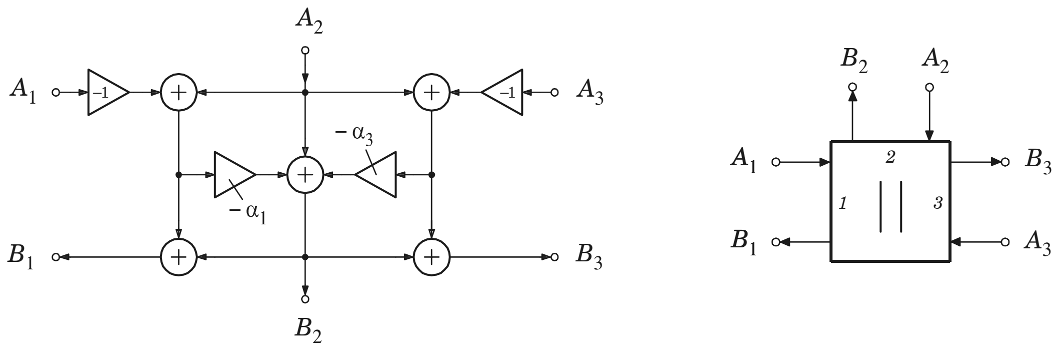

Figure 5: Possible structure for realizing the Ŝ-matrix of Equation 31 and its commonly used symbol.

then we get

$$\alpha_3 = 1 \tag{33}$$

and we find that B_3 becomes independent of A_3,
as can be seen from

$$\hat{\mathbf{S}} = \begin{bmatrix} \alpha_1 - 1 & 1 - \alpha_1 & 1 \\ \alpha_1 & -\alpha_1 & 1 \\ \alpha_1 & 1 - \alpha_1 & 0 \end{bmatrix} \tag{34}$$

In this case, we obtain a Three-Port Parallel adaptor with *reflection-free* Port 3 and Port 2 being dependent, which can be indicated as a *constrained* three-port parallel adaptor.
Figure 6 shows a possible realization and its commonly used symbol.

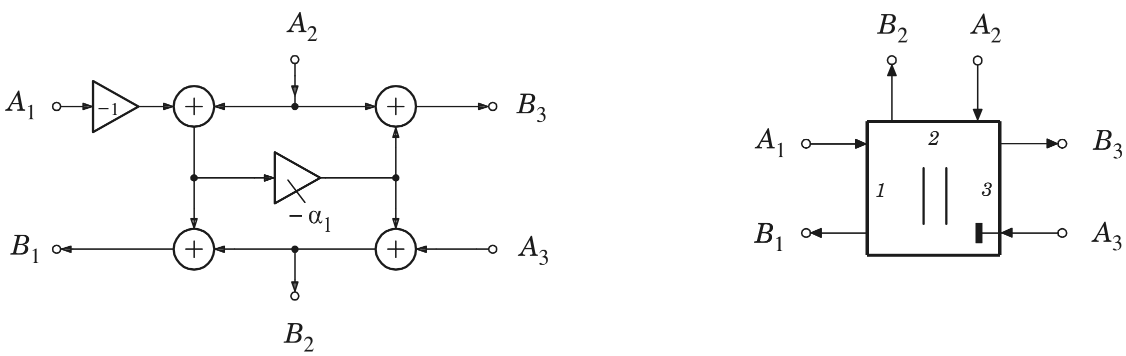

Figure 6: Possible structure for realizing the Ŝ-matrix of Equation 34 and the symbol to represents it with.

3 The Two-Port Adaptor

Given the equations directly following from Figure 7

$$U_1 = U_2 \quad \text{and} \quad I_1 + I_2 = 0 \tag{35}$$

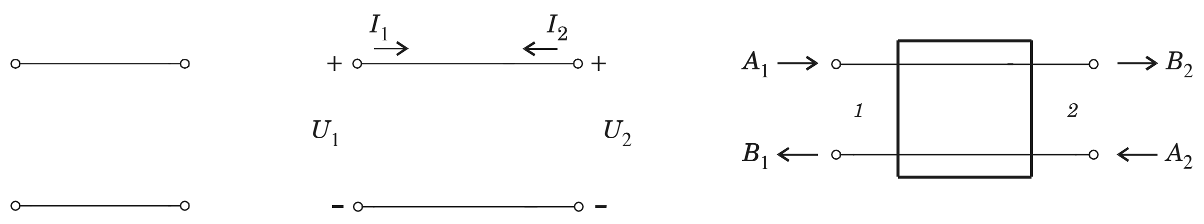

Figure 7: Definitions with respect to the two-port (parallel) adapter.

and the definitions of the voltage wave variables

$$A_k = U_k + I_k R_k \tag{36a}$$
$$B_k = U_k - I_k R_k \tag{36b}$$

we find

$$U_k = \frac{A_k + B_k}{2} \quad \text{and} \quad I_k = \frac{A_k - B_k}{2R_k} = 2g_k\left(A_k - B_k\right) \quad \text{with } k = 1, 2 \tag{37}$$

While $g_k = \dfrac{1}{R_k}$, we can perform the following derivations by combining equations (35) and (37)

$$g_1\left(A_1 - B_1\right) + g_2\left(A_2 - B_2\right) = 0 \tag{38}$$

$$A_1 + B_1 = A_2 + B_2 \tag{39}$$

So, if we eliminate B_2

$$-B_2 = -A_1 - B_1 + A_2 \tag{40}$$

we can express B_1 in terms of the A-inputs

$$B_1 = \frac{g_1 - g_2}{g_1 + g_2} A_1 + \frac{2g_2}{g_1 + g_2} A_2 \tag{41}$$

If we define

$$\alpha_k = \frac{2g_k}{g_1 + g_2}$$

then (24) reduces to

$$B_1 = (\alpha_1 - 1)\ A_1 + \alpha_2 A_2 \tag{42}$$

More generally

$$B_k = \alpha_1 A_1 + \alpha_2 A_2 - A_k \tag{43}$$

It also follows that

$$\alpha_1 + \alpha_2 = 2 \qquad \text{while} \quad 0 \leq \alpha_k \leq 2 \tag{44}$$

We can write the wave equations in matrix notation: $\mathbf{B} = \hat{\mathbf{S}} \cdot \mathbf{A}$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 - 1 & \alpha_2 \\ \alpha_1 & \alpha_2 - 1 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{45}$$

If we now eliminate α_2 by realizing (see equation (44)) that

$$\alpha_2 = 2 - \alpha_1$$

we obtain for the matrix $\hat{\mathbf{S}}$

$$\hat{\mathbf{S}} = \begin{bmatrix} \alpha_1 - 1 & 2 - \alpha_1 \\ \alpha_1 & 1 - \alpha_1 \end{bmatrix} \tag{46}$$

Such an adaptor can be denoted as a Two-Port adaptor. A possible realization using only one (constant coëfficiënt) multiplier, 3 adders and one sign-inversion is given in Figure 8a.

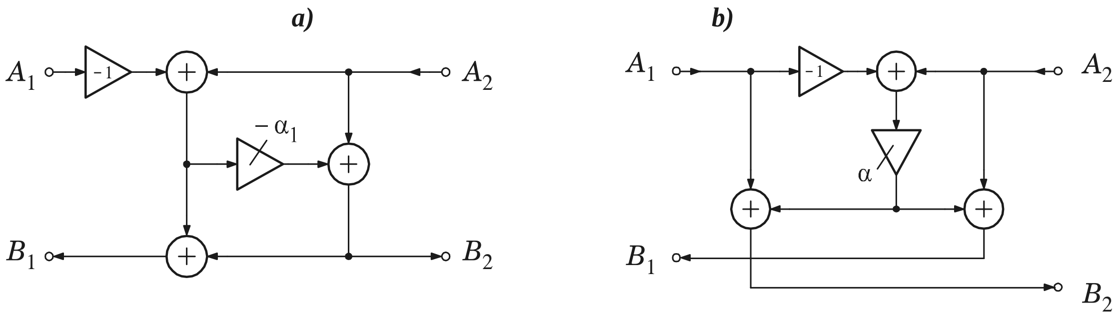

Figure 8: Possible structures for realizing *a)* the Ŝ-matrix of Equation 46**, and *b)* the Ŝ-matrix of Equation** 48**.**

We can introduce a new variable α, with $\alpha = 1 - \alpha_1$, for which can be derived that

$$\alpha = \frac{R_1 - R_2}{R_1 + R_2} \tag{47}$$

to find an alternative $\hat{\mathbf{S}}$ matrix:

$$\hat{\mathbf{S}} = \begin{bmatrix} -\alpha & 1+\alpha \\ 1-\alpha & \alpha \end{bmatrix} \tag{48}$$

with a possible realization shown in Figure 8b.

Figure 9 shows two symbols to identify a two-port with. Figure 9a is the most widely used, but we prefer to use the simple version of 9b in our MATLAB drawings.

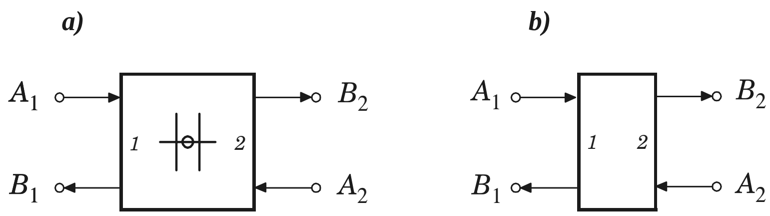

Figure 9: Two symbols to be used for a two-port adaptor in block diagrams.